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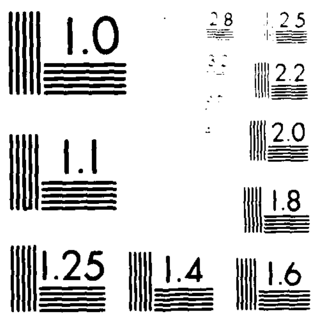

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AN ANALYSIS OF THE p-VERSION OF THE FINITE ELEMENT METHOD FOR NEARLY INCOMPRESSIBLE MATERIALS. UNIFORMLY VALID, OPTIMAL ERROR ESTIMATES.

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ABSTRACT

In this paper we analyze the behavior of the so-called p-version of the finite element method when applied to the equations of plane strain linear elasticity. We establish optimal rate error estimates that are uniformly valid, independent of the value of the Poisson ratio, ν, in the interval $]0,1/2[$. This shows that the p-version <u>does not</u> exhibit the degeneracy phenomenon which has led to the use of various, only partially justified techniques of reduced integration or mixed formulations for more standard finite element schemes and the case of a nearly incompressible material.

AMS (MOS) Subject Classifications: 65N30, 73K25

Key Words: Finite element method, optimal (uniformly valid) error estimates, linear elasticity, nearly incompressible materials

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AN ANALYSIS OF THE p-VERSION OF THE FINITE ELEMENT METHOD FOR NEARLY INCOMPRESSIBLE MATERIALS. UNIFORMLY VALID, OPTIMAL ERROR ESTIMATES.

Michael Vogelius*

1. Introduction

It is well known that standard finite element schemes based on C^0 elements and direct minimization of the displacement energy have very poor convergence properties for nearly incompressible materials. The same degeneracy occurs in connection with plate-bending models that account for shear, such as the Timoshenko plate (or beam) model, when the thickness of the structure is small (cf. [17]). In order to get reliable results one would have to take a mesh size h which in the first case is much less than $1-2\nu$ (ν denotes the Poisson ratio) and in the second case much less than the thickness of the structure. This is in most cases not practically feasible.

One way to try to avoid the latter problem is to include more regular elements, e.g. C^1 (cf. [11]), but in more than one dimension the extra continuity conditions are very difficult to handle in practice.

By far the most simple and successful remedy in both cases has been the technique of "reduced integration" (cf. [18]). The idea is to replace some of the integrals in the energy by the corresponding expressions arising from the application of some quadrature rule. The finite element spaces remain simple C^0, for example, piecewise linear. This approach of course is only a partial

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remedy since it cannot provide better results measured in terms of energy, but it makes it possible to obtain good approximations to the individual components of the displacement. In many instances there is a close connection between this approach of reduced integration and a so-called mixed formulation of the variational problem (cf. [2,14]). There are numerous papers in the engineering literature devoted to the questions of what terms in the energy to evaluate only approximately and what quadrature rule to use for a particular element. It is quite clear however that no comprehensive mathematical analysis exists at this time to justify the benefits of reduced integration. Some partial justification may be found e.g. in [2], for the case of the Timoshenko beam model and small thickness, and in [12] for the case of one particular element and either an almost incompressible material or the Timoshenko plate model.

Finite element methods in general are based on some partition of the domain (typically a triangulation) and function spaces whose elements are piecewise polynomials on each triangle. The h-version of the finite element method (the one discussed so far) obtains convergent solutions by letting the mesh size h (i.e., the diameter of the largest triangle) tend to 0 using only piecewise polynomials of some fixed degree. In contrast the so-called p-version of the finite element method is based on a fixed triangulation and the degree of the piecewise polynomials is increased in order to give convergence. For more details about the convergence properties of the p-version and combinations of the two versions see [5,6,7].

In this paper we study the behavior of the p-version for an almost incompressible material. Recent numerical experiments, reported in [6], strongly indicate that the degeneracy which has led to the use of such techniques as reduced integration is not present in the case of the p-version. The main theorem of this paper, Theorem 3.1, verifies that conjecture.

We shall briefly illustrate the previous discussion with one of the computations from [6]. Figures 1 and 2 show the relative error in energy versus the number of degrees of freedom (on a log-log scale) for an "edge cracked square panel". The mathematical model used is that of 2-dimensional plane strain elasticity. In Fig. 1 the Poisson ratio is 0.3 and in Fig. 2 it is 0.4999 (i.e., the material is almost incompressible). For the h-version the functions are C^0 piecewise linear, and the p-version is based on C^0 piecewise polynomials on the triangular mesh shown in the upper left hand corner. By comparison of the two figures it is evident that the p-version is quite insensitive to changes in the Poisson ratio, whereas the h-version for $\nu = 0.4999$ does not seem convergent at all (to be more precise, one is not in the asymptotic range even with 500 degrees of freedom).

A remark concerning the convergence rates. In Fig. 1 it is seen that the p-version has a convergence rate which is twice that of the h-version. This is a consequence of the fact that polynomials are well-suited to approximate a wide class of singular solutions to elliptic boundary value problems

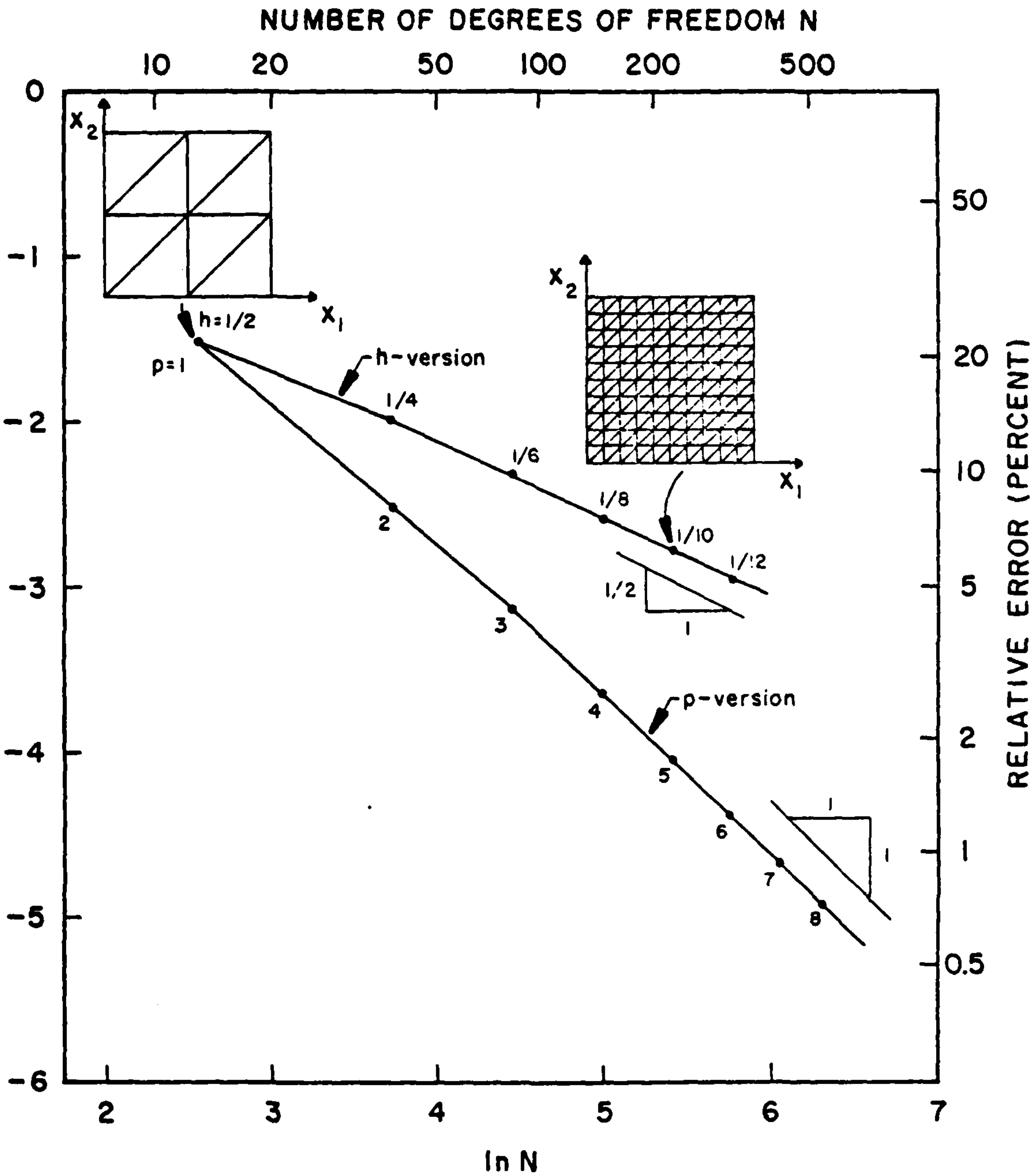

Fig. 1. Relative error in strain energy vs. number of degrees of freedom.
Poisson ratio: 0.3.
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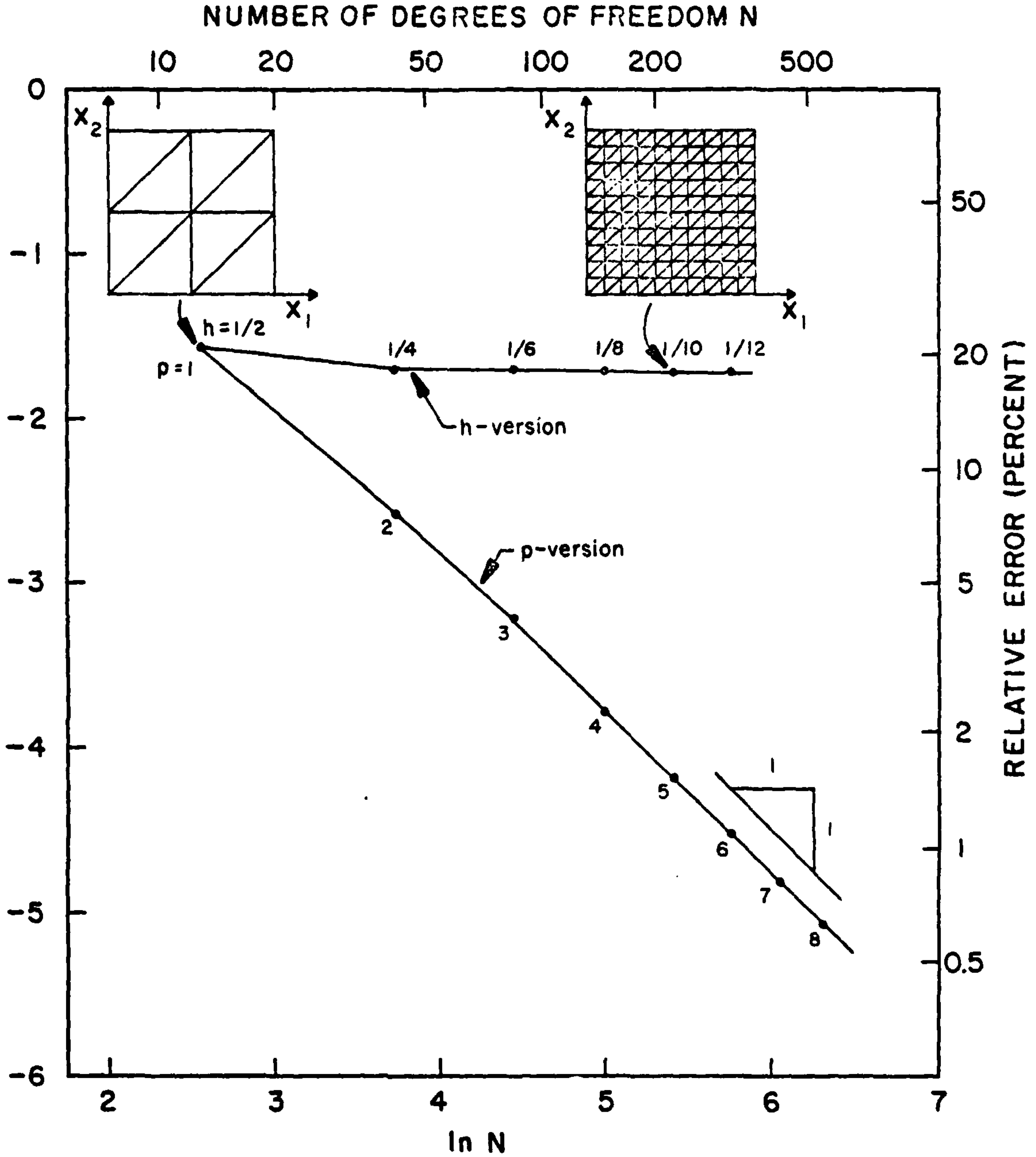

Fig. 2. Relative error in strain energy vs. number of degrees of freedom.
Poisson ratio: 0.4999.
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as the degree increases (for more details see [7]). The convergence rates shown in Fig. 1 are governed by the singularity of the solution at the crack tip.

The analysis presented in this paper does not study the sensitivity of this "corner singularity" convergence rate to changes in ν. Rather the results are directed at the case when the convergence rate is determined by singularities of the solution inside the domain (see remarks following Theorem 3.1).

Numerical evidence (as the one shown here) suggests that the "corner singularity" convergence rate is also insensitive to changes in ν whenever the "corner singularity" itself does not change with ν. In some cases however (cf. [6]) the "corner singularity" depends on ν, and therefore naturally also the convergence rate of the p-version.

The contents of this paper are as follows. In section 2 we give a precise statement of the continuous problem to be studied in this paper and a general stability analysis for a wide range of approximate solutions. For V in a large class of closed subspaces of $H^1(\Omega)$ we assign a number $C(V)$ that measures the stability in a sense made precise by Lemma 2.1. In section 3 we establish uniformly valid, optimal rate error estimates for the p-version. The proof of the main theorem, Theorem 3.1, relies on a result concerning the invertibility of the divergence operator as established in [16]. In the appendix we prove a uniformly valid a priori estimate for the continuous problem which is needed for the proof of Theorem 3.1.

An analysis similar to that presented in this paper could also be carried out for the Timoshenko plate model.

Throughout this paper $H^k(\Omega)$ denotes the standard Sobolev space of functions that have all derivatives of order $\leq k$ in $L^2(\Omega)$. Whenever no ambiguity is possible the same notation shall also be used for vector valued functions whose components are in $H^k(\Omega)$. $H^{-k}(\Omega)$ denotes the dual of $H^k(\Omega)$ (and not of $\mathring{H}^k(\Omega)$ as sometimes used). Spaces with noninteger exponents are defined by complex interpolation (cf. [9,13]). The norm on the space $H^s(\Omega)$ is denoted $\|\cdot\|_{s,\Omega}$. The same conventions apply tc Sobolev spaces on the smooth boundary $\partial\Omega$.

Acknowledgement

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2. Formulation of the continuous problem and a general stability analysis.

The problem which we shall study is that of plane strain linear elasticity on a bounded domain $\Omega \subseteq \mathbb{R}^2$ with a C^∞ boundary $\partial\Omega$. Formulated in terms of displacements $\underline{u}_\nu = (u_{\nu,1}, u_{\nu,2})$ the continuous problem is

$$(1) \qquad -\Delta \underline{u}_\nu - \frac{1}{1-2\nu} \nabla(\nabla \cdot \underline{u}_\nu) = \underline{F} \text{ in } \Omega,$$

where $\underline{F}$ represents an external force (to be specific $\underline{F}$ is the external force scaled by $2(1+\nu)/E$). $0 < \nu < \frac{1}{2}$ denotes the Poisson ratio and $0 < E$ the Young modulus; a value of ν close to $\frac{1}{2}$ corresponds to a nearly incompressible material. For boundary conditions we consider the natural (stress) boundary conditions associated with (1); these may be written as

$$(2) \qquad \sum_{j=1}^{2} \left(\varepsilon_{ij}(\underline{u}_\nu) + \delta_{ij} \frac{\nu}{1-2\nu} \nabla \cdot \underline{u}_\nu\right) n_j = g_i, \quad 1 \le i \le 2 \text{ on } \partial\Omega,$$

where $\{n_j\}_{j=1}^2$ is the unit outward normal to $\partial\Omega$, $\{\varepsilon_{ij}\}_{i,j=1}^2$ is the strain tensor, given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} u_j + \frac{\partial}{\partial x_j} u_i\right)$$

and $\{g_i\}_{i=1}^2$ represents the force being applied at $\partial\Omega$ (here scaled by $(1+\nu)/E$). The boundary value problem (1) and (2) has a solution iff

$$(3) \qquad \iint_\Omega \underline{F}\,\underline{R}\,d\underline{x} + 2\int_{\partial\Omega} \underline{g}\,\underline{R}\,ds = 0$$

for any rigid motion $\underline{R}$.

In the following it shall always be implicitly assumed that $\underline{F}$ and $\underline{g}$ satisfy this relation. The solution, of course, is only determined up to a rigid motion.

Let $\mathcal{H}^1(\Omega)$ denote $(H^1(\Omega))^2/\{\text{rigid motions}\}$ with the standard norm for a quotient space (also denoted $\|\cdot\|_{1,\Omega}$). Solving the boundary value problem (1) and (2) is equivalent to minimizing the energy expression

$$(4) \qquad \iint_\Omega \left[\sum_{i,j=1}^{2} (\varepsilon_{ij}(\underline{v}))^2 + \frac{\nu}{1-2\nu} (\nabla\cdot\underline{v})^2\right] d\underline{x} - \iint_\Omega \underline{F}\,\underline{v}\,d\underline{x} - 2\int_{\partial\Omega} \underline{g}\,\underline{v}\,ds$$

in $(H^1(\Omega))^2$ (or $\mathcal{H}^1(\Omega)$ to make the solution unique). Let V be a closed subspace of $\mathcal{H}^1(\Omega)$ (not necessarily proper) and let $\underline{u}_{\nu,V}$ denote the minimizer for (4) in V. For $V = \mathcal{H}^1(\Omega)$ we drop the subscript V to be consistent with earlier notation. The specific application we have in mind regarding numerical procedures is of course to let V be some space of (relatively high degree) piecewise polynomials on some triangulation of Ω.

In this section we establish an estimate on $\underline{u}_\nu - \underline{u}_{\nu,V}$ which is valid uniformly in ν (the nontrivial and interesting case being when ν approaches $\frac{1}{2}$). For reasons of exposition we have chosen to give this result in a general form here and then devote the next section to the application for the p-version of the finite element method.

If we introduce the space $\mathcal{D} = \nabla\cdot V = \{\nabla\cdot\underline{v} \mid \underline{v} \in V\}$ and Lagrangian multipliers then it is quite clear that $\underline{u}_{\nu,V}$ may also be characterized as the first component of the vector $(\underline{v},r,s) \in V \times \mathcal{D} \times \mathcal{D}$ which satisfies

$$(5) \quad B_\nu\big((\underline{v},r,s);\ (\underline{w},\phi,\psi)\big) = \iint_\Omega \underline{F}\ \underline{w}\ d\underline{x} + 2\int_{\partial\Omega} \underline{g}\ \underline{w}\ ds$$

$$\forall(\underline{w},\phi,\psi) \in V \times \mathcal{D} \times \mathcal{D}\ .$$

Here B_ν denotes the symmetric bilinear form

$$(6) \quad 2\iint_\Omega \sum_{i,j=1}^{2} \varepsilon_{ij}(\underline{v})\ \varepsilon_{ij}(\underline{w})\ d\underline{x} + \iint_\Omega r\phi\ d\underline{x}$$

$$+ \iint_\Omega s(\delta\phi-\nabla\cdot\underline{w})\ d\underline{x} + \iint_\Omega (\delta r-\nabla\cdot\underline{v})\psi\ d\underline{x}$$

with δ given by $\delta = \left(\frac{1-2\nu}{2\nu}\right)^{1/2}$. (The second component of the solution to (5) is $r = \left(\frac{2\nu}{1-2\nu}\right)^{1/2} \nabla\cdot\underline{u}_{\nu,V}$ and the third is $s = \frac{-2\nu}{1-2\nu}\nabla\cdot\underline{u}_{\nu,V}$).

In the following we shall always assume that $\mathcal{D} \neq 0$ and $\mathcal{D}$ is closed in $L^2(\Omega)$; from the closed graph theorem we then get that there exists a constant C such that given any $r \in \mathcal{D}$ one can find $\underline{v} \in V$ with

$$(7) \quad \begin{aligned} &\nabla\cdot\underline{v} = r \quad \text{and} \\ &\|\underline{v}\|_{1,\Omega} \leq C\|r\|_{0,\Omega}\ . \end{aligned}$$

Let $C(V)$ denote the infimum of all constants C that can be used

with the construction (7). As a tool for our later analysis we prove the following stability estimate for the form B_ν.

Lemma 2.1. Let B_ν denote the bilinear form defined in (6). Let V be a closed subspace of $H^1(\Omega)$ with the property that $\mathcal{D} = \nabla\cdot V \neq 0$ is closed in $L^2(\Omega)$.

There exists a constant $d > 0$, independent of $\nu \in]0,1/2[$ and V, such that

$$\inf_{(\underline{v},r,s)\in S} \ \sup_{(\underline{w},\phi,\psi)\in S} |B_\nu((\underline{v},r,s);\ (\underline{w},\phi,\psi))| \geq d\, C(V)^{-2}$$

where $S = \{(\underline{v},r,s) \in V \times \mathcal{D} \times \mathcal{D} \mid \|\underline{v}\|_{1,\Omega} + \|r\|_{0,\Omega} + \|s\|_{0,\Omega} = 1\}$.

Proof: One argues as in [4]; we give just a sketch of a proof here in order to ensure that the statement about the specific form of the lower bound is true.

From Korn's inequality (cf. [10])

$$(8)\quad B_\nu((\underline{v},r,s);\ (\underline{v},r,-s)) \geq d(\|\underline{v}\|^2_{1,\Omega} + \|r\|^2_{0,\Omega}).$$

d in the following takes the role of a generic (small) constant that does not depend on ν or V. If $\underline{w}$ is selected such that $\nabla\cdot\underline{w} = -s$ and $\|\underline{w}\|_{1,\Omega} \leq 2C(V)\|s\|_{0,\Omega}$ (this can be accomplished according to the definition of $C(V)$) then

$$(9)\quad B_\nu((\underline{v},r,s);\ (\underline{w},0,0)) \geq \|s\|^2_{0,\Omega} - d^{-1}C(V)\|\underline{v}\|_{1,\Omega}\|s\|_{0,\Omega}$$
$$\geq \frac{1}{2}\|s\|^2_{0,\Omega} - d^{-1}C(V)^2\|\underline{v}\|^2_{1,\Omega}.$$

Taking an appropriate linear combination of $(\underline{v},r,-s)$ and $(\underline{w},0,0)$ we obtain, due to (8) and (9), a $(\underline{w},\phi,\psi)$ that satisfies the estimates

$$B_\nu((\underline{v},r,s);\ (\underline{w},\phi,\psi)) \geq d\left(\|\underline{v}\|^2_{1,\Omega} + \|r\|^2_{0,\Omega} + C(V)^{-2}\|s\|^2_{0,\Omega}\right)$$

(10)

$$\|\underline{w}\|_{1,\Omega} + \|\phi\|_{0,\Omega} + \|\psi\|_{0,\Omega} \leq \|\underline{v}\|_{1,\Omega} + \|r\|_{0,\Omega} + (1+C(V)^{-1})\|s\|_{0,\Omega} .$$

It is easy to see that $C(V)$ is bounded away from 0 independently of V, and that the estimates (10) therefore lead to the desired result. □

Based on Lemma 2.1 we now prove an estimate for $\underline{u}_\nu - \underline{u}_{\nu,V}$.

<u>Lemma</u> 2.2. Let $\underline{u}_\nu$ be the minimizer of (4) in $H^1(\Omega)$ and $\underline{u}_{\nu,V}$ the corresponding minimizer in the closed subspace V. Assume that the space $D = \nabla\cdot V \neq 0$ is closed in $L^2(\Omega)$ and let $C(V)$ denote the constant defined previously. There exists a constant D, independent of $\nu \in]0,1/2[$ and V, such that

$$\|\underline{u}_\nu - \underline{u}_{\nu,V}\|_{1,\Omega} + \frac{1}{1-2\nu}\|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,V})\|_{0,\Omega}$$
$$\leq D\, C(V)^2 \left(\inf_{\underline{v}\in V}\|\underline{u}_\nu - \underline{v}\|_{1,\Omega} + \inf_{w\in D}\left\|\frac{1}{1-2\nu}\nabla\cdot\underline{u}_\nu - w\right\|_{0,\Omega}\right)$$

<u>Proof</u>: This estimate is trivial as long as ν does not get close to 1/2. It is therefore sufficient to consider $1/4 < \nu < 1/2$. The symmetric bilinear form B_ν introduced before satisfies

(11) $$|B_\nu((\underline{v},r,s);(\underline{w},\phi,\psi))| \leq D\left(\|\underline{v}\|_{1,\Omega} + \|r\|_{0,\Omega} + \|s\|_{0,\Omega}\right) \times \left(\|\underline{w}\|_{1,\Omega} + \|\phi\|_{0,\Omega} + \|\psi\|_{0,\Omega}\right)$$

for ν in this interval. It is not difficult to see (cf. [3]) that Lemma 2.1 combined with (11) leads to the inequality

$$\|\underline{u}_\nu - \underline{u}_{\nu,V}\|_{1,\Omega} + \left(\frac{2\nu}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,V})\|_{0,\Omega} + \frac{2\nu}{1-2\nu}\|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,V})\|_{0,\Omega}$$

$$\leq (1+Dd^{-1}C(V)^2)\Big\{\inf_{\underline{v}\in V}\|\underline{u}_\nu - \underline{v}\|_{1,\Omega} + \inf_{w\in \mathcal{D}}\|\left(\frac{2\nu}{1-2\nu}\right)^{1/2}\nabla\cdot\underline{u}_\nu - w\|_{0,\Omega}$$

$$+ \inf_{w\in \mathcal{D}}\|\frac{2\nu}{1-2\nu}\nabla\cdot\underline{u}_\nu - w\|_{0,\Omega}\Big\} ,$$

from which the desired estimate follows immediately. □

3. The p-version F.E.M.: uniform, optimal error estimates.

In this section we study the convergence properties of approximate solutions to (1) and (2) that are obtained by minimizing the energy expression (4) over a family of finite element subspaces associated with the p-version.

Let $\Sigma = \{T_i\}_{i=1}^{M}$ be a triangulation of the domain Ω. We assume that no vertex of a triangle lies in the interior of a side of another. Triangles of Σ may possibly have one curved side (in common with $\partial\Omega$). We assume that any of the triangles with one curved side can be mapped affinely onto a Lipschitz domain of the form

$$\{(x_1,x_2) \mid 0 < x_1 < 1,\ 0 < x_2 < f(x_1)\}$$

where f is monotone and satisfies $f(0) = 1$, $f(1) = 0$.

Let $P^{[p],0}$ denote the set of continuous functions whose restriction to each triangle is a polynomial of degree at most p. $P^{[p],-1}$ denotes the set of functions, whose restriction to each triangle is given by a polynomial of degree at most p (no inter-element continuity requirements), with the additional property that:

R.1. "At any internal vertex (x_1^0,x_2^0) of the triangulation, where four triangles meet as the intersection of two straight lines, the corresponding values $\{Q_i^p(x_1^0,x_2^0)\}_{i=1}^{4}$ satisfy

$$\sum_{i\ \mathrm{even}} Q_i^p(x_1^0,x_2^0) = \sum_{i\ \mathrm{odd}} Q_i^p(x_1^0,x_2^0)\text{."}$$

(cf. Fig. 1 in [16]).

The space $(P^{[p],0})^2/\{\text{rigid motions}\}$ is also for short denoted $(P^{[p],0})^2$, $p \geq 1$.

For the proof of the main theorem in this section we need the following 3 auxiliary results.

Lemma 3.1. Let $C((P^{[p],0})^2)$ denote the constant introduced in section 2. There exist constants C and K, independent of p, such that

$$C((P^{[p],0})^2) \le C\, p^K$$

for any $p \ge 1$.

Proof: This follows directly from Theorem 2.1 and Remark 2.2 in [16].

Lemma 3.2. Let $k \ge 1$. There exists a constant C_k such that

$$\inf_{\underline{Q}^p \in (P^{[p],0})^2} \|\underline{v} - \underline{Q}^p\|_{1,\Omega} \le C_k\, p^{-k+1} \|\underline{v}\|_{k,\Omega}$$

for any $p \ge 1$ and any $\underline{v} \in (H^k(\Omega))^2$.

Proof: See [7].

Lemma 3.3. Let $k \ge 0$. There exists a constant C_k such that for any $p \ge 1$ and any $w \in H^k(\Omega)$,

$$\inf_{Q^p} \|w - Q^p\|_{0,\Omega} \le C_k p^{-k} \|w\|_{k,\Omega},$$

where the infimum is taken over $Q^p \in \nabla\cdot((P^{[p],0})^2)$.

Proof: In [16] it is shown that $\nabla\cdot((P^{[p],0})^2) = P^{[p-1],-1}$ for p sufficiently large. Due to the inclusion $P^{[p-1],0} \subseteq P^{[p-1],-1}$ the estimate follows immediately from [7] (as did Lemma 3.2).

We are now in a position to prove the main result of this paper, a theorem that provides optimal error estimates for the p-version of the finite element method based on direct minimization of the energy in displacement form (4). The important feature, that distinguishes these estimates from the optimal ones that hold for the more standard h-version, is that they are uniform in ν. We shall give a discussion to clarify in what sense these estimates are optimal immediately following the proof.

Theorem 3.1. Assume that $s \geq -1$ and $\nu \in]0,1/2[$. Let $\underline{u}_\nu$ denote the solution to (1) and (2). For any $p \geq 1$ let $\underline{u}_{\nu,p}$ denote the element (or rather the equivalence class) in $(P^{[p],0})^2$ that minimizes (4). Given $\varepsilon > 0$ there exists $C_{s,\varepsilon}$ (depending on s, ε but independent of ν and p) such that whenever $\underline{F} \in H^s(\Omega)$ and $\underline{g} \in H^{s+1/2}(\partial\Omega)$

$$\begin{aligned} &\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,p})\|_{0,\Omega} \\ (14) \qquad &\leq C_{s,\varepsilon}\, p^{-s-1+\varepsilon}\left(\|\underline{F}\|_{s,\Omega} + \|\underline{g}\|_{s+1/2,\partial\Omega}\right) \end{aligned}$$

Proof: By Lemmas 2.2 and 3.1

$$\begin{aligned} &\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \frac{1}{1-2\nu} \|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,p})\|_{0,\Omega} \\ (15) \qquad &\leq C\, p^{2K}\left(\inf_{\underline{v}} \|\underline{u}_\nu - \underline{v}\|_{1,\Omega} + \inf_{w} \left\|\frac{1}{1-2\nu}\nabla\cdot\underline{u}_\nu - w\right\|_{0,\Omega}\right), \end{aligned}$$

where the first infimum is taken over $\underline{v} \in (P^{[p],0})^2$ and the second over $w \in \nabla\cdot((P^{[p],0})^2)$.

An application of Lemmas 3.2 and 3.3 show that the right hand side is bounded by

$$C_N\, p^{2K-N-1}\left(\|\underline{u}_\nu\|_{N+2,\Omega} + \frac{1}{1-2\nu}\|\nabla\cdot\underline{u}_\nu\|_{N+1,\Omega}\right),$$

and because of the uniform a priori estimates given in Theorem A.1 of the appendix, this is at most

$$C_N\, p^{2K-N-1}\left(\|\underline{F}\|_{N,\Omega} + \|\underline{g}\|_{N+1/2,\partial\Omega}\right).$$

If we replace $1/(1-2\nu)$ in the left hand side of (15) by $(1/(1-2\nu))^{1/2}$ (which is certainly smaller) we conclude that

$$\begin{aligned}&\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2}\|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,p})\|_{0,\Omega}\\ &\le C_N p^{2K-N-1}\left(\|\underline{F}\|_{N,\Omega} + \|\underline{g}\|_{N+1/2,\partial\Omega}\right).\end{aligned} \tag{16}$$

Since $\underline{u}_{\nu,p}$ can also be interpreted as the projection of $\underline{u}_\nu$ onto $(P^{[p],0})^2$ in the energy inner-product associated with (4),

$$\begin{aligned}&\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2}\|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,p})\|_{0,\Omega}\\ &\le C\left(\|\underline{u}_\nu\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2}\|\nabla\cdot\underline{u}_\nu\|_{0,\Omega}\right)\\ &\le C\left(\|\underline{F}\|_{-1,\Omega} + \|\underline{g}\|_{-1/2,\partial\Omega}\right)\end{aligned} \tag{17}$$

(the last inequality is due to Theorem A.1). Interpolating between (16) and (17) subject to the finite dimensional constraint that

$$\iint_{\Omega} \underline{F}\ \underline{R}\ dx\ dy + 2 \int_{\partial\Omega} \underline{g}\ \underline{R}\ ds = 0$$

for any rigid motion $\underline{R}$ (cf. Theorems 12.5 and 13.3 of [13]) we get that

$$(18)\quad \begin{aligned} &\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,p})\|_{0,\Omega} \\ &\le C_{N,\theta}\, p^{\theta(2K-N-1)} \left(\|\underline{F}\|_{(N+1)\theta-1,\Omega} + \|\underline{g}\|_{(N+1)\theta-1/2,\partial\Omega}\right) . \end{aligned}$$

Choose N larger than $2K(s+1)/\varepsilon$ and define θ to be $(s+1)/(N+1)$. This ensures that $2K\theta < \varepsilon$, and from the inequality (18) we thus get

$$\begin{aligned} &\|\underline{u}_\nu - \underline{u}_{\nu,p}\|_{1,\Omega} + \left(\frac{1}{1-2\nu}\right)^{1/2} \|\nabla\cdot(\underline{u}_\nu - \underline{u}_{\nu,p})\|_{0,\Omega} \\ &\le C_{s,\varepsilon}\, p^{-s-1+\varepsilon} \left(\|\underline{F}\|_{s,\Omega} + \|\underline{g}\|_{s+1/2,\partial\Omega}\right) , \end{aligned}$$

exactly as desired. □

Remark 3.1. The estimate (14) is optimal in the following sense: If there exists ν_0 and a sequence $\underline{v}_j \in (P^{[p_j],0})^2$, $p_{j+1}/p_j \le \Lambda < \infty$, $p_j \uparrow \infty$, such that

$$\|\underline{u}_{\nu_0} - \underline{v}_j\|_{1,\Omega} \le C\, p_j^{-s-1-\delta} ,$$

for some $\delta > 0$, then one may conclude (cf. [7]) that

$$\underline{u}_{\nu_0} \in H^{s+2+t}_{loc}(T_i)$$

for any $t < \delta$ and any triangle $T_i \in \Sigma$ ($\underline{u}_{\nu_0} \in H^{s+2+t}_{loc}(T_i)$ means that $\underline{u}_{\nu_0} \in H^{s+2+t}(K)$ on all compact subsets K of T_i). This immediately implies that

$$\underline{F} \in H^{s+t}_{loc}(T_i) \tag{19}$$

for any triangle $T_i \in \Sigma$. For a general $\underline{F} \in H^s(\Omega)$, (19) is obviously not satisfied and the best possible convergence rate is therefore p^{-s-1}.

For a particular class of singularities typically associated with corners of the domain Ω it has been verified that one does better than (14), indeed the p-version converges at twice the rate of the h-version finite element method (cf. [7]). As already observed, numerical evidence indicates that this improved convergence rate is also insensitive to changes in ν. This claim however is not verified by the present analysis.

Remark 3.2. The norm on the left hand side of (14) is equivalent to the energy norm

$$\left(\iint_\Omega \left[\sum_{i,j=1}^{2} (\varepsilon_{ij}(\underline{v}))^2 + \frac{\nu}{1-2\nu} (\nabla\cdot\underline{v})^2 \right] d\underline{x} \right)^{1/2}$$

with constants that are independent of ν. Let W_ν denote the energy expression given in (4). From the observation about equivalence of norms and the estimate (14) it follows that

$$0 \leq W_\nu(\underline{u}_{\nu,p})-W_\nu(\underline{u}_\nu) \leq C_{k,\varepsilon}\, p^{-2k-2+\varepsilon}(\|\underline{F}\|^2_{k,\Omega} + \|\underline{g}\|^2_{k+1/2,\partial\Omega}) .$$

The situation is as already mentioned quite different for the h-version of the finite element method. If the minimization is performed over all continuous piecewise polynomials of degree $\leq$ k+1 on a triangulation Σ_h of mesh size h, then we have the estimate

$$(20) \quad 0 \leq W_\nu(\underline{u}^h_\nu)-W_\nu(\underline{u}_\nu) \leq C_{k,\nu}\, h^{2k+2}(\|\underline{F}\|^2_{k,\Omega} + \|\underline{g}\|^2_{k+1/2,\partial\Omega})$$

for any fixed value of $\nu \in]0,1/2[$. The estimate (20) however is not valid with a constant C_k that is independent of ν; rather $C_{k,\nu}$ tends to ∞ as ν approaches 1/2. Numerical evidence (e.g. the figures shown in [15]) suggests that the best possible estimate is

$$0 \leq W_\nu(\underline{u}^h_\nu) - W_\nu(\underline{u}_\nu) \leq C_k h^{2k}$$

with C_k independent of ν. That is to say, the uniformly valid convergence rates for the energy are reduced by a factor of h^{-2} from the optimal ones. Similar statements seem to hold for the convergence rates for the displacements.

Appendix. Uniform a priori estimates for the continuous problem

Because of the regularity properties of solutions to elliptic boundary value problems on smooth domains it is clear that $\underline{u}_\nu$, the solution to (1) and (2), is an element of $H^{k+2}(\Omega)$ provided $\underline{F} \in H^k(\Omega)$ and $\underset{\sim}{g} \in H^{k+1/2}(\partial\Omega)$. In this appendix we shall prove that the norm of $\underline{u}_\nu$ in $H^{k+2}(\Omega)$ and the norm of $\frac{1}{1-2\nu}\nabla\cdot\underline{u}_\nu$ in $H^{k+1}(\Omega)$ stay bounded independent of ν (this is exactly the result which was used in the proof of Theorem 3.1). This result may seem surprising at first, since (1) and (2) looks like a singular perturbation problem for ν close to 1/2; what we therefore in effect show is that there are no boundary layers.

Let us introduce the following auxiliary problem

$$(21)\qquad \begin{aligned} -\nu\,\Delta\underline{U}_\nu + \nabla P_\nu &= \tilde{\underline{F}} \quad \text{in } \Omega\,, \\ \nabla\cdot\underline{U}_\nu &= \tilde{G} \quad \text{in } \Omega \end{aligned}$$

with boundary conditions

$$(22)\qquad \sum_{j=1}^{2} \varepsilon_{ij}(\underline{U}_\nu)\, n_j - P_\nu n_i = \tilde{g}_i\,,\quad 1 \le i \le 2 \quad \text{on } \partial\Omega.$$

This is a sort of generalized Stokes problem; we now proceed to establish apriori estimates for $\underline{U}_\nu$ and P_ν in terms of $\tilde{\underline{F}}$, $\tilde{G}$ and $\tilde{\underline{g}}$. In doing so we rely on the paper of Agmon, Douglis and Nirenberg [1]. Our proof therefore consists merely of checking that the system (21), (22) falls into the general class studied in that paper.

<u>Lemma</u> A.1. Assume that $1/4 < \nu < 1/2$ and that k is an integer ≥ 0. If $\underline{U}_\nu \in H^2(\Omega)$, $P_\nu \in H^1(\Omega)$ denotes a solution to (21) and (22) for data $\underline{\tilde{F}} \in H^k(\Omega)$, $\tilde{G} \in H^{k+1}(\Omega)$ and $\underline{\tilde{g}} \in H^{k+1/2}(\partial\Omega)$ then $\underline{U}_\nu \in H^{k+2}(\Omega)$, $P_\nu \in H^{k+1}(\Omega)$ and

$$\|\underline{U}_\nu\|_{k+2,\Omega} + \|P_\nu\|_{k+1,\Omega} \leq C_k\big(\|\underline{\tilde{F}}\|_{k,\Omega} + \|\tilde{G}\|_{k+1,\Omega} + \|\underline{\tilde{g}}\|_{k+1/2,\partial\Omega} + \|\underline{U}_\nu\|_{0,\Omega} + \|P_\nu\|_{0,\Omega}\big) ,$$

with C_k independent of ν.

<u>Proof</u>: If we denote the vector $(\underline{U}_\nu, P_\nu)$ by $\underline{V}_\nu$ then (21) may be written

$$\begin{bmatrix} -\nu\Delta & \nabla \\ \nabla\cdot & 0 \end{bmatrix} \underline{V}_\nu = \begin{bmatrix} \underline{\tilde{F}} \\ \tilde{G} \end{bmatrix} .$$

The symbol of this system is

$$\underline{\underline{\ell}}(\underline{\xi}) = \begin{bmatrix} -\nu|\underline{\xi}|^2 & 0 & \xi_1 \\ 0 & -\nu|\underline{\xi}|^2 & \xi_2 \\ \xi_1 & \xi_2 & 0 \end{bmatrix}$$

By defining $t_j = 2$, $j = 1,2$, $t_3 = 1$, $s_i = 0$, $i = 1,2$, and $s_3 = -1$ we obtain

$$\deg(\ell_{ij}(\underline{\xi})) \leq s_i + t_j .$$

Since in this proof we follow the notation of [1] exactly, we shall not care to explain the meaning of these indices but rather refer the reader to that paper for more details. The principal part $\ell'_{ij}(\underline{\xi})$ is

equal to $\ell_{ij}(\underline{\xi})$, and we compute

$$L(\underline{\xi}) = \det\,(\ell'_{ij}(\underline{\xi})) = \nu|\underline{\xi}|^4 ,$$

which ensures the uniform ellipticity of the system (condition 1.7 on page 39 of [1]).

If $\underline{\xi}$ and $\underline{\xi}'$ are two nonzero real vectors then the equation

$$L(\underline{\xi} + \tau\underline{\xi}') = 0 \tag{23}$$

has the roots

$$\tau_0^{\pm}(\underline{\xi},\underline{\xi}') = \frac{-\underline{\xi}\cdot\underline{\xi}' \pm i\sqrt{|\underline{\xi}|^2|\underline{\xi}'|^2 - (\underline{\xi}\cdot\underline{\xi}')^2}}{|\underline{\xi}'|^2}$$

both with multiplicity 2. If $\underline{\xi}$ and $\underline{\xi}'$ are linearly independent then $|\underline{\xi}|^2|\underline{\xi}'|^2 - (\underline{\xi}\cdot\underline{\xi}')^2 > 0$, and we conclude that the equation (23) in this case has exactly 2 roots (counted according to multiplicity) with positive imaginary part. This verifies the supplementary condition on page 39 of [1].

The boundary condition (22) may be represented by the symbol

$$\underline{\underline{B}}(Q,\underline{\xi}) = \begin{bmatrix} \xi_1 n_1 + \frac{1}{2}\xi_2 n_2 & \frac{1}{2}\xi_1 n_2 & -n_1 \\ \frac{1}{2}\xi_2 n_1 & \xi_2 n_2 + \frac{1}{2}\xi_1 n_1 & -n_2 \end{bmatrix}$$

(The unit normal $\underline{n}$ depends on Q, the particular point at the boundary of Ω.)

By choosing $r_h = -1$, $h = 1,2$, we obtain the inequalities

$$\deg(B_{hj}(Q,\underline{\xi})) \le r_h + t_j ,$$

and since these actually are identities we conclude that the principal part $\underline{\underline{B}}'$ is identical to $\underline{\underline{B}}$. The product $\underline{\underline{B}}\,(Q,\underline{\xi})\cdot\underline{\underline{\ell}}(\underline{\xi})$ is

$$\begin{bmatrix} -\nu|\underline{\xi}|^2(\xi_1 n_1+\frac{1}{2}\,\xi_2 n_2)-\xi_1 n_1 & -\frac{\nu}{2}\,|\underline{\xi}|^2\,\xi_1 n_2-\xi_2 n_1 & \xi_1^2 n_1+\xi_1\xi_2 n_2 \\ -\frac{\nu}{2}\,|\underline{\xi}|^2\xi_2 n_1-\xi_1 n_2 & -\nu|\underline{\xi}|^2(\xi_2 n_2+\frac{1}{2}\,\xi_1 n_1)-\xi_2 n_2 & \xi_2^2 n_2+\xi_1\xi_2 n_1 \end{bmatrix}$$

For the complementing boundary condition (p. 42 of [1]) to be satisfied we have to check that every nontrivial linear combination of the two rows in the matrix

$$\underline{\underline{B}}\,(Q,\underline{t}+\tau\underline{n})\cdot\underline{\underline{\ell}}(\underline{t}+\tau\underline{n})$$

is nonzero modulo the polynomial

$$M^+(Q,\underline{t},\tau) = (\tau-\tau_0^+(\underline{t},\underline{n}))^2 .$$

This has to be satisfied for any tangent vector $\underline{t} \neq 0$.

If we consider $C_1 \times$ (first row) + $C_2 \times$ (last row) the third component has the form

$$(\underline{C}\cdot\underline{t} + \tau\,\underline{C}\cdot\underline{n})\,\tau$$

where $\underline{C}$ is the vector (C_1,C_2). This can be 0 modulo $M^+(Q,\underline{t},\tau)$ only if $\underline{C}\cdot\underline{t} = \underline{C}\cdot\underline{n} = 0$, i.e., iff $\underline{C} = 0$. We have therefore verified the complementing boundary condition.

Since ν is always in the interval]1/4,1/2[(i.e., the lower bound in the uniform ellipticity estimate does not degenerate) the desired estimate follows directly

from Theorem 10.5 of [1]. □

Based on Lemmas 2.1 and A.1 we now prove the following apriori estimates for $\underline{u}_\nu$, the solution to (1) and (2). The important feature is that the constants C_k are independent of $\nu \in]0,1/2[$.

Theorem A.1. Let k be an integer ≥ -1. If $\underline{u}_\nu \in H^1(\Omega)$ denotes the solution to (1) and (2) for data $\underline{F} \in H^k(\Omega)$, $\underline{g} \in H^{k+1/2}(\partial\Omega)$ then $\underline{u}_\nu \in H^{k+2}(\Omega)$ and

$$(24) \quad \|\underline{u}_\nu\|_{k+2,\Omega} + \frac{1}{1-2\nu}\|\nabla\cdot\underline{u}_\nu\|_{k+1,\Omega} \leq C_k(\|\underline{F}\|_{k,\Omega} + \|\underline{g}\|_{k+1/2,\partial\Omega})$$

with C_k independent of ν.

Note: The solution to (1) and (2) is only uniquely determined up to a rigid motion (i.e., in $H^1(\Omega)$). The first norm on the left side in (24) refers to the norm of the corresponding equivalence class.

Proof: That $\underline{u}_\nu \in H^{k+2}(\Omega)$ provided $\underline{F} \in H^k(\Omega)$ and $\underline{g} \in H^{k+1/2}(\partial\Omega)$ follows directly from standard regularity results for elliptic boundary value problems on smooth domains. That the estimate (24) is uniform in ν for $\nu \in]0,c[$, $c < 1/2$, is also clear from that theory. We shall now show that (24) is valid uniformly in $\nu \in]c,1/2[$, for an appropriate choice of $c > 1/4$.

In the case $k = -1$ this result follows immediately from Lemma 2.1, by taking $V = H^1(\Omega)$. To be more specific

$$\|\underset{\sim}{u}_\nu\|_{1,\Omega} + \frac{2\nu}{1-2\nu}\|\nabla\cdot\underset{\sim}{u}_\nu\|_{0,\Omega}$$

$$\leq d^{-1}C(H^1)^2 \sup_{(\underset{\sim}{w},\phi,\psi)\in S} |B_\nu((\underset{\sim}{u}_\nu,r,s);\ (\underset{\sim}{w},\phi,\psi))|$$

with $r = \left(\frac{2\nu}{1-2\nu}\right)^{1/2}\nabla\cdot\underset{\sim}{u}_\nu$ and $s = \frac{-2\nu}{1-2\nu}\nabla\cdot\underset{\sim}{u}_\nu$. The right hand side is according to (5) equal to

$$d^{-1}C(H^1)^2 \left|\iint_\Omega \underset{\sim}{F}\,\underset{\sim}{w}\,d\underset{\sim}{x} + 2\int_{\partial\Omega} \underset{\sim}{g}\,\underset{\sim}{w}\,ds\right| ,$$

and this gives the desired estimate.

In the case $k \geq 0$ let us introduce

$$\underset{\sim}{U}_\nu = \underset{\sim}{u}_\nu$$

$$P_\nu = -\frac{\nu}{1-2\nu}\nabla\cdot\underset{\sim}{u}_\nu ,$$

then $\underset{\sim}{U}_\nu \in H^2(\Omega)$, $P_\nu \in H^1(\Omega)$ satisfies (21) and (22) with $\tilde{\underset{\sim}{F}} = \nu\underset{\sim}{F}$, $\tilde{G} = \nabla\cdot\underset{\sim}{u}_\nu$ and $\tilde{\underset{\sim}{g}} = \underset{\sim}{g}$. By Lemma A.1,

$$\|\underset{\sim}{u}_\nu\|_{k+2,\Omega} + \frac{\nu}{1-2\nu}\|\nabla\cdot\underset{\sim}{u}_\nu\|_{k+1,\Omega}$$

$$(25) \quad = \|\underset{\sim}{U}_\nu\|_{k+2,\Omega} + \|P_\nu\|_{k+1,\Omega}$$

$$\leq C_k\left(\nu\|\underset{\sim}{F}\|_{k,\Omega} + \|\nabla\cdot\underset{\sim}{u}_\nu\|_{k+1,\Omega} + \|\underset{\sim}{g}\|_{k+1/2,\partial\Omega} + \|\underset{\sim}{u}_\nu\|_{0,\Omega} + \frac{\nu}{1-2\nu}\|\nabla\cdot\underset{\sim}{u}_\nu\|_{0,\Omega}\right).$$

The last two terms on the right hand side can by an application of this theorem with $k=-1$ (which has already been proven)

be estimated by

$$C_k(\|\underline{F}\|_{-1,\Omega} + \|\underline{g}\|_{-1/2,\partial\Omega}) ,$$

and therefore

$$\|\underline{u}_\nu\|_{k+2,\Omega} + \frac{\nu}{1-2\nu} \|\nabla\cdot\underline{u}_\nu\|_{k+1,\Omega}$$

$$\leq C_k(\|\underline{F}\|_{k,\Omega} + \|\nabla\cdot\underline{u}_\nu\|_{k+1,\Omega} + \|\underline{g}\|_{k+1/2,\partial\Omega}) .$$

Since $\nu \in]c,1/2[$, we may by choosing c sufficiently close to 1/2 arrange that $C_k < \frac{1}{2} \cdot \frac{\nu}{1-2\nu}$, and thus absorb the term involving $\nabla\cdot\underline{u}_\nu$ on the right side into the corresponding term on the left. This concludes the proof of the theorem. □

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In this paper we analyze the behavior of the so-called p-version of the finite element method when applied to the equations of plane strain linear elasticity. We establish optimal rate error estimates that are uniformly valid, independent of the value of the Poisson ratio, ν, in the interval $]0,1/2[$. This shows that the p-version does not exhibit the degeneracy phenomenon which has led to the use of various, only partially justified techniques of reduced integration or mixed formulations for more standard finite element schemes and the case of a nearly incompressible material.

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